Journal of Engineering Mathematics **26**: 363–377, 1992.

Temperature distribution in a bi-material body with a line of cracks under uniform heat flow

S.I. CHOU
Department of Engineering Mechanics, University of Nebraska-Lincoln, Lincoln, NE 68588-0347, U.S.A.

Received 14 November 1990; accepted in revised form 27 September 1991

Abstract. The temperature distribution in a bi-material body with a line of cracks at the interface under uniform heat flow is solved. The boundary value problem is reduced to the solution of a singular integral equation of Cauchy type whose solution is given by Muskhelishvili. Temperature distributions for the case of a single crack, and for the case of two collinear cracks are given.

1. Introduction

The problem of finding the temperature distribution in an infinite medium with a line of cracks under uniform heat flow has been considered by Sih [1]. He reduced the solution of the boundary value problem to the solution of the Hilbert problem [2]. Temperature distribution and heat flux in the infinite medium with a single crack, with two collinear cracks, and infinite collinear cracks were given. It was shown that near the crack tip the heat flux exhibits the inverse square-root singularity in terms of the radial distance from the crack tip, while the temperature remains bounded. As observed by Sih [1], the problem was motivated by the need for the temperature distribution in a body with cracks in fracture mechanics when temperature effects are considered. When a uniform heat flow is disturbed by the cracks in the body, there occurs a local intensification of the temperature gradient near the crack tip. Associated with this is the rise in thermal stress near the crack tip which is often the cause of failure. The problem is thus of practical interest.

In this paper, the problem of temperature distribution in a bi-material body with a line of insulated cracks at the interface under uniform heat flow is considered. The temperature distribution in a bi-material body without a line of cracks is first determined. The boundary value problem under consideration is then reduced to the problem of determining the temperature distribution in the bi-material body where heat flux is prescribed at the crack surfaces. The Fourier transform is used in the analysis. The problem is reduced to the solution of a singular integral equation of Cauchy type which can be solved by the method of Muskhelishvili [2]. Temperature distributions for the case of a single crack, and for the case of two collinear cracks are given as examples. It is shown that the results reduce to those of Sih [1] when the bi-material body is reduced to the single material body.

2. Temperature distribution in the bi-material body without cracks

Consider an infinite bi-material body consisting of region Ω_1 in the upper plane $y > 0$ and region Ω_2 in the lower plane $y < 0$ with coefficients of thermal conductivity κ_1 and κ_2,

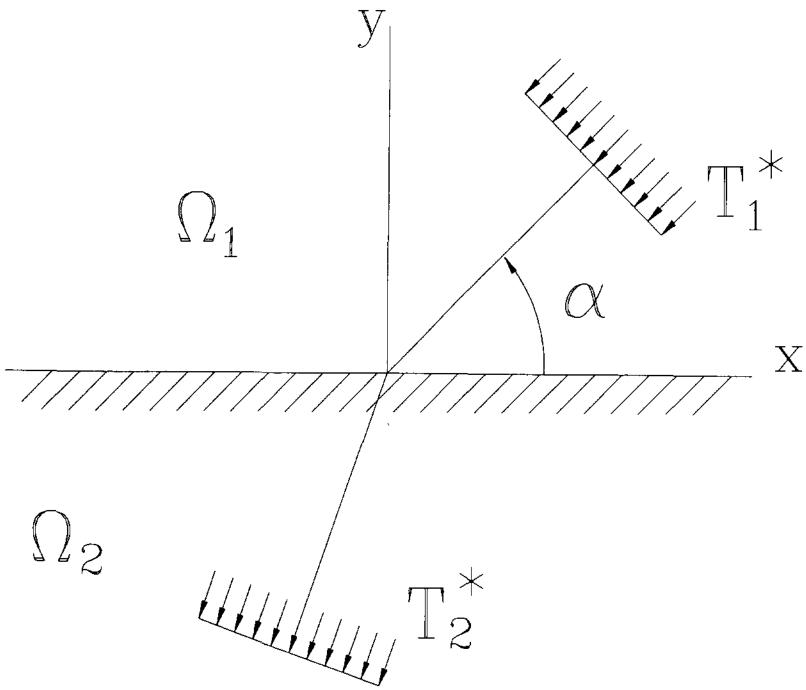

Fig. 1. Infinite bi-material body without cracks.

respectively, Fig. 1. The interface of the regions is perfectly bonded and heat can be transferred through it.

Let the temperature distribution in the region Ω_1 be

$$T_1^* = q(x \cos \alpha + y \sin \alpha)\,, \tag{1}$$

where q is the uniform temperature gradient and let the temperature distribution in Ω_2 be given as

$$T_2^* = a_1 x + a_2 y\,. \tag{2}$$

These satisfy the two dimensional Laplace equation

$$\nabla^2 T = 0\,. \tag{3}$$

The constants a_1 and a_2 are determined from the continuity of the temperature at the interface $y = 0$ given by

$$T_1^* = T_2^* \tag{4}$$

and the continuity of the heat flux at the interface given by

$$\kappa_1 \frac{\partial T_1^*}{\partial y} = \kappa_2 \frac{\partial T_2^*}{\partial y} \tag{5}$$

which yield

$$a_1 = q \cos\alpha\ , \qquad a_2 = \frac{\kappa_1}{\kappa_2}\, q \sin\alpha\ .$$

The temperature distribution in Ω_2, $y<0$ is then given by

$$T_2^* = q\left(x \cos\alpha + \frac{\kappa_1}{\kappa_2}\, y \sin\alpha\right). \tag{6}$$

3. Formulation of the problem

Let the interface of the regions $y=0$ have a line of insulated cracks $L = L_1 + L_2 + \cdots + L_p$, Fig. 2. At infinity, region Ω_1 has temperature distribution, T_1^* given by Eq. (1), and region Ω_2 then has the temperature distribution T_2^* given by Eq. (6).

Let the temperature distribution in region Ω_i be T_i', $i = 1, 2$ which must satisfy the Laplace equation

$$\nabla^2 f_i = 0 \tag{7}$$

with $f_i = T_i'$ and the following conditions:

$$T_1'(x, 0+) = T_2'(x, 0-)\ , \quad x \notin L\ , \tag{8}$$

$$\kappa_1 \frac{\partial T_1'(x, 0+)}{\partial y} = \kappa_2 \frac{\partial T_2'(x, 0-)}{\partial y}\ , \quad x \notin L\ , \tag{9}$$

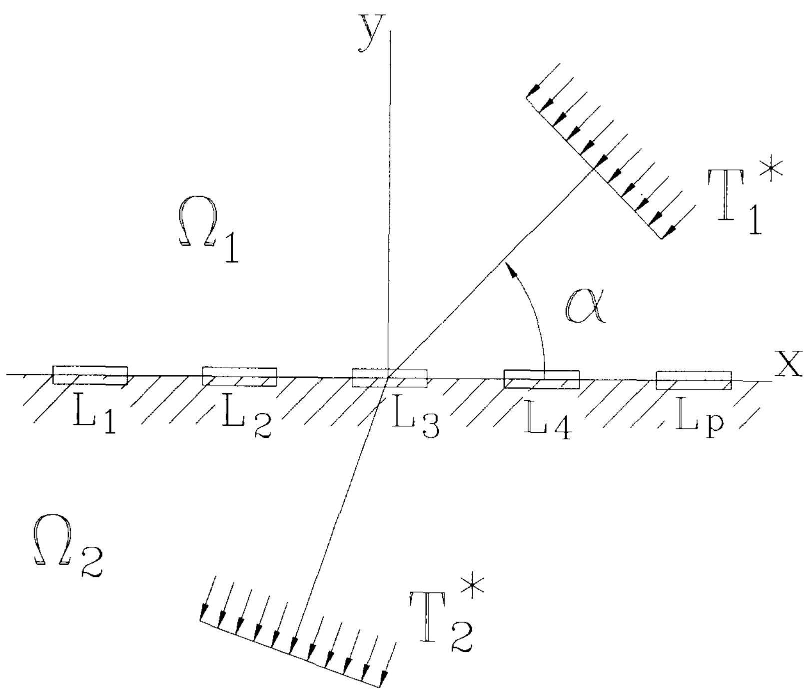

Fig. 2. Infinite bi-material body with a line of cracks.

$$\frac{\partial T_1'(x,0+)}{\partial y} = \frac{\partial T_2'(x,0-)}{\partial y} = 0\,, \quad x \in L\,, \tag{10}$$

$$T_1' - T_1^* \to 0\,, \qquad T_2' - T_2^* \to 0 \quad \text{as } x^2 + y^2 \to \infty\,. \tag{11}$$

In the above, (8) and (9) are the continuity conditions for the temperature and the temperature gradient at the interface, respectively; (10) expresses the fact that the crack surfaces are insulated; (11) shows that the temperature in the body away from the cracks approaches that of the uncracked body.

Let $T_i' = T_i + T_i^*$, $i = 1, 2$, where T_i is the disturbance temperature due to the presence of the cracks, and is assumed to be bounded near the crack tips and to vanish at infinity. It follows that T_i satisfies the Laplace equation (7) in Ω_i. The conditions (8), (9), (10), and (11), respectively, become

$$T_1(x,0+) = T_2(x,0-)\,, \quad x \notin L\,, \tag{12}$$

$$\kappa_1 \frac{\partial T_1(x,0+)}{\partial y} = \kappa_2 \frac{\partial T_2(x,0-)}{\partial y}\,, \quad x \notin L\,, \tag{13}$$

$$\frac{\partial T_1(x,0+)}{\partial y} = -q \sin\alpha\,, \quad x \in L\,, \tag{14}$$

$$\frac{\partial T_2(x,0-)}{\partial y} = -\frac{\kappa_1}{\kappa_2} q \sin\alpha\,, \quad x \in L\,, \tag{15}$$

$$T_1 \to 0\,, \quad T_2 \to 0 \text{ at infinity.} \tag{16}$$

Conditions (14) and (15) yield

$$\kappa_1 \frac{\partial T_1(x,0+)}{\partial y} = \kappa_2 \frac{\partial T_2(x,0-)}{\partial y}\,, \quad x \in L\,,$$

which together with (13) imply

$$\kappa_1 \frac{\partial T_1(x,0+)}{\partial y} = \kappa_2 \frac{\partial T_2(x,0-)}{\partial y} \quad \text{for } y = 0 \text{ and all } x\,. \tag{17}$$

4. Solution of the problem

Introduce the Fourier transform

$$\tilde{f}(\xi, y) = \int_{-\infty}^{\infty} f(\xi, y)\, e^{i\xi x}\, dx$$

and its inverse

$$f(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\xi, y)\, e^{-i\xi x}\, d\xi\,.$$

Application of the Fourier transform [3] to the Laplace equation (7) yields

$$\left(\frac{d^2}{dy^2} - \xi^2\right)\hat{f}(\xi, y) = 0 ,$$

which has the solution

$$\tilde{f}(\xi, y) = A_1(\xi)\, e^{-|\xi| y} + B_1(\xi)\, e^{|\xi| y} .$$

Condition (16) is satisfied if one chooses

$$\tilde{T}_1(\xi, y) = A(\xi)\, e^{-|\xi| y} , \tag{18}$$

$$\tilde{T}_2(\xi, y) = B(\xi)\, e^{|\xi| y} \tag{19}$$

and obtains

$$T_1(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\xi)\, e^{-|\xi| y - ix\xi}\, d\xi ,$$

$$T_2(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\xi)\, e^{|\xi| y - ix\xi}\, d\xi .$$

Substituting these into condition (17), one obtains

$$B(\xi) = -\frac{\kappa_1}{\kappa_2} A(\xi) \tag{20}$$

and finally

$$T_1(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\xi)\, e^{-|\xi| y - ix\xi}\, d\xi , \tag{21}$$

$$T_2(x, y) = -\frac{\kappa_1}{\kappa_2}\frac{1}{2\pi}\int_{-\infty}^{\infty} A(\xi)\, e^{|\xi| y - ix\xi}\, d\xi . \tag{22}$$

Condition (12) gives

$$\left(1 + \frac{\kappa_1}{\kappa_2}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty} A(\xi)\, e^{-ix\xi}\, d\xi = \begin{cases} 0 , & x \not\in L \\ \Delta T(x) , & x \in L \end{cases} , \tag{23}$$

where

$$\Delta T(x) = T_1(x, 0^+) - T_2(x, 0^-) .$$

It follows from (23) that

$$\left(1 + \frac{\kappa_1}{\kappa_2}\right) A(\xi) = \int_L \Delta T(x)\, e^{ix\xi}\, dx , \quad x \in L . \tag{24}$$

Substituting (24) into (21) and (22) gives

$$T_1(x, y) = \frac{\kappa_2}{\kappa_1 + \kappa_2}\frac{1}{2\pi}\int_L \Delta T(t)\, dt \int_{-\infty}^{\infty} e^{-|\xi||y| - i(x-t)\xi}\, d\xi ,$$

$$T_2(x, y) = \frac{-\kappa_1}{\kappa_1 + \kappa_2} \frac{1}{2\pi} \int_L \Delta T(t)\, dt \int_{-\infty}^{\infty} e^{-|\xi||y| - i(x-t)\xi}\, d\xi .$$

The second integral in these equations can be integrated and, as a result, one obtains

$$T_1(x, y) = \frac{\kappa_2}{\kappa_1 + \kappa_2} \frac{y}{\pi} \int_L \frac{\Delta T(t)}{(x-t)^2 + y^2}\, dt, \quad y > 0, \tag{25}$$

$$T_2(x, y) = \frac{\kappa_1}{\kappa_1 + \kappa_2} \frac{y}{\pi} \int_L \frac{\Delta T(t)}{(x-t)^2 + y^2}\, dt, \quad y < 0. \tag{26}$$

An alternative solution of the problem leading to these equations without the use of Fourier transform is given in Appendix 1.

It remains to determine the temperature difference ΔT over the cracks from either of the conditions (14) or (15). For this, let

$$T_1(x, y) = \frac{2\kappa_2}{\kappa_1 + \kappa_2} \operatorname{Re} \Phi(z), \quad y > 0, \tag{27}$$

$$T_2(x, y) = \frac{2\kappa_1}{\kappa_1 + \kappa_2} \operatorname{Re} \Phi(z), \quad y < 0. \tag{28}$$

where $z = x + iy$ and $\Phi(z)$ is defined by the Cauchy integral

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\Delta T(t)}{t - z}\, dt. \tag{29}$$

Since $\Phi(z)$ is analytic outside L, the Cauchy–Riemann equation

$$\frac{\partial}{\partial y} \operatorname{Re} \Phi(z) = -\frac{\partial}{\partial x} \operatorname{Im} \Phi(z)$$

holds. The boundary condition (14) then yields

$$\frac{\partial}{\partial x} \operatorname{Im} \Phi(x + i0) = \frac{\kappa_1 + \kappa_2}{2\kappa_2} q \sin\alpha, \quad x \in L. \tag{30}$$

Here, the limit value $\Phi(x + i0)$, $x \in L$, is found from the Plemelj's formulas for Cauchy integral (29) [2, §17]. Integration of the boundary condition (30) with respect to x yields the following:

$$\frac{1}{\pi i} \int_L \frac{\Delta T(t)}{t - x}\, dt = i \frac{\kappa_1 + \kappa_2}{\kappa_2} xq \sin\alpha + C_j, \quad x, t \in L, \tag{31}$$

where the constant C_j applies when $x \in L_j$, $j = 1, 2, \ldots, p$, and the integral on the left stands for the Cauchy principal value.

Equation (31) is a singular integral equation of Cauchy type whose solution is given by Muskhelishvili [2, §88–§90]. The solution of (31), which is bounded at the crack tips, can be written as follows:

$$\Delta T(x)=\frac{[R(x)]^{1/2}}{\pi\mathrm{i}}\int_L \frac{f(t)\,\mathrm{d}t}{[R(t)]^{1/2}(t-x)}+\frac{[R(x)]^{1/2}}{\pi\mathrm{i}}\sum_{j=1}^{p} C_j\int_{L_j}\frac{\mathrm{d}t}{[R(t)]^{1/2}(t-x)}\,,\quad x\in L\,, \tag{32}$$

where

$$f(t)=\frac{\mathrm{i}(\kappa_1+\kappa_2)}{\kappa_2}\,tq\sin\alpha$$

and

$$R(z)=\prod_{j=1}^{p}(z-a_j)(z-b_j)\,.$$

Here a_j and b_j denote the end points of the crack L_j and $[R(t)]^{1/2}$ is understood to be the limiting value of $[R(z)]^{1/2}$ as $z\to t\in L$ in $\mathrm{Im}\,z>0$. The constants C_j satisfy the equations

$$\sum_{n=1}^{p} a_{mn}C_n+A_m=0\quad (m=0,1,2,\dots,p-1)\,, \tag{33}$$

where

$$a_{mn}=\int_{L_n}\frac{t^m\,\mathrm{d}t}{[R(t)]^{1/2}}\,,\qquad A_m=\int_L\frac{t^m f(t)\,\mathrm{d}t}{[R(t)]^{1/2}}\,. \tag{34}$$

With ΔT found from (32), the temperature distribution in both regions can be found from (25) and (26).

5. The problem of a single crack

Consider the problem of a single crack $|x|<a$ at the interface. For this problem, letting $p=1$ in (32) and taking $[R(t)]^{1/2}=\mathrm{i}(a^2-t^2)^{1/2}$, Eq. (32) becomes

$$\begin{aligned}\Delta T(x)=&\frac{q\sin\alpha}{\pi}\,\frac{\kappa_1+\kappa_2}{\kappa_2}\,(a^2-x^2)^{1/2}\int_{-a}^{a}\frac{t\,\mathrm{d}t}{(a^2-t^2)^{1/2}(t-x)}\\&+C_1\frac{(a^2-x^2)^{1/2}}{\pi\mathrm{i}}\int_{-a}^{a}\frac{\mathrm{d}t}{(a^2-t^2)^{1/2}(t-x)}\,,\quad |x|<a\,.\end{aligned} \tag{35}$$

Using the results [4, form. 15.2 (21)]

$$\int_{-a}^{a}\frac{\mathrm{d}t}{(a^2-t^2)^{1/2}(x-t)}=\begin{cases}\pi\,\mathrm{sgn}(x)/(x^2-a^2)^{1/2} & |x|>a\\ 0 & |x|<a\end{cases}\,.$$

Equation (35) becomes

$$\Delta T(x)=q\sin\alpha\,\frac{\kappa_1+\kappa_2}{\kappa_2}\,(a^2-x^2)^{1/2}\,,\quad |x|<a\,. \tag{36}$$

Equations (27) and (28) then yield

$$T_1(x, y) = 2q \sin\alpha \operatorname{Re}\Phi_1(z), \quad y > 0, \tag{37}$$

$$T_2(x, y) = 2q \sin\alpha \frac{\kappa_1}{\kappa_2} \operatorname{Re}\Phi_1(z), \quad y < 0, \tag{38}$$

where

$$\Phi_1(z) = \frac{1}{2\pi \mathrm{i}} \int_{-a}^{a} \frac{(a^2 - t^2)^{1/2}}{t - z}\, \mathrm{d}t. \tag{39}$$

It is shown in Appendix 2 that

$$\Phi_1(z) = \frac{1}{2\mathrm{i}} [(z^2 - a^2)^{1/2} - z], \tag{40}$$

and finally

$$T_1(x, y) = -qy \sin\alpha + \frac{q \sin\alpha}{\sqrt{2}} \{[(x^2 - y^2 - a^2)^2 + 4x^2y^2]^{1/2} - (x^2 - y^2 - a^2)\}^{1/2}, \quad y > 0, \tag{41}$$

$$T_2(x, y) = -\frac{\kappa_1}{\kappa_2} qy \sin\alpha - \frac{\kappa_1}{\kappa_2} \frac{q \sin\alpha}{\sqrt{2}} \{[(x^2 - y^2 - a^2)^2 + 4x^2y^2]^{1/2} - (x^2 - y^2 - a^2)\}^{1/2}, \quad y < 0. \tag{42}$$

The temperature distribution in the body becomes

$$T_1'(x, y) = qx \cos\alpha + \frac{q \sin\alpha}{\sqrt{2}} \{[(x^2 - y^2 - a^2)^2 + 4x^2y^2]^{1/2} - (x^2 - y^2 - a^2)\}^{1/2}, \quad y > 0, \tag{43}$$

$$T_2'(x, y) = qx \cos\alpha - \frac{\kappa_1}{\kappa_2} \frac{q \sin\alpha}{\sqrt{2}} \{[(x^2 - y^2 - a^2)^2 + 4x^2y^2]^{1/2} - (x^2 - y^2 - a^2)\}^{1/2}, \quad y < 0. \tag{44}$$

It is easily seen that these reduce to that given by Sih [1] if one sets $\kappa_1 = \kappa_2$. Note that the first term on the right hand side of Sih's expression [1, form. (22)] must be multiplied by a factor $\operatorname{sgn}(y)$.

6. The problem of two collinear cracks

Let there be two collinear cracks $a < |x| < b$ at the interface $y = 0$. For this problem, in (32), take $p = 2$ and

$$[R(x)]^{1/2} = \mathrm{i}[(b^2 - x^2)(x^2 - a^2)]^{1/2} \operatorname{sgn}(x), \quad a < |x| < b.$$

For this case, the first term in (32) involving the integral

$$\left(\int_{-b}^{-a} + \int_{a}^{b}\right) \frac{t\,dt}{[R(t)]^{1/2}(t-x)} = -2ix \int_{a}^{b} \frac{t\,dt}{[(b^2-t^2)(t^2-a^2)]^{1/2}(t^2-x^2)}\,, \quad a<|x|<b$$

is shown to vanish by using the relation [4, form. 15.2 (32)]

$$\int_{a}^{b} \frac{t\,dt}{[(b^2-t^2)(t^2-a^2)]^{1/2}(t^2-x^2)} = \begin{cases} \pi/2[(b^2-x^2)(a^2-x^2)]^{-1/2} & |x|<a \\ 0 & a<|x|<b \\ -\pi/2[(x^2-b^2)(x^2-a^2)]^{-1/2} & |x|>b\,. \end{cases}$$

The constants C_1 and C_2 are found to be

$$C_1 = -C_2 = i\,\frac{\kappa_1+\kappa_2}{\kappa_2}\,\frac{\pi bq \sin\alpha}{2K}$$

and the second term in (32) now takes the form

$$\sum_{j=1}^{2} C_j \int_{L_j} \frac{dt}{[R(t)]^{1/2}(t-x)} = 2iC_1 x \int_{a}^{b} \frac{dt}{[(b^2-t^2)(t^2-a^2)]^{1/2}(t^2-x^2)}\,. \tag{45}$$

The integral on the right-hand side can be found from [5, form. 218.02, 415.01] as

$$\int_{a}^{b} \frac{dt}{[(b^2-t^2)(t^2-a^2)]^{1/2}(t^2-x^2)} = -\frac{KZ(A(x),k)}{|x|[(b^2-x^2)(x^2-a^2)]^{1/2}}\,, \quad a<|x|<b\,,$$

where $Z(A,k)$ is the Jacobian zeta function defined by [5, form. 140.01]

$$Z(A,k) = E(A,k) - \frac{E}{K}\,F(A,k)$$

and K and E denote the complete elliptic integrals of the first and second kind, respectively [5, form. 110.06, 110.07], while $F(A,k)$ and $E(A,k)$ with

$$k = \frac{(b^2-a^2)^{1/2}}{b}\,, \qquad \sin A(x) = \left(\frac{b^2-x^2}{b^2-a^2}\right)^{1/2}$$

are the Legendre's incomplete elliptic integrals of the first and second kind, respectively [5, form. 110.02, 110.03].

Finally one obtains

$$\Delta T(x) = \frac{\kappa_1+\kappa_2}{\kappa_2}\,bq\sin\alpha\, Z(A,k)\,, \quad a<|x|<b\,. \tag{46}$$

By letting $a \to 0$, i.e. letting $k \to 1$, and using the relations [5, form. 111.05]

$$E(1) = 1\,, \qquad K(1) = \infty\,,$$

and [5, form. 111.04]

$$E(A,1) = \sin A\,,$$

the case of a single crack of length $2b$ is recovered. Equation (46) takes the form

$$\Delta T(x) = q \sin \alpha \, \frac{\kappa_1 + \kappa_2}{\kappa_2} (b^2 - x^2)^{1/2}, \quad |x| < b \,,$$

which is seen to be that given in (36), as it should be.

From (27) and (28) with $\Delta T(x)$ given by (46), one can write

$$T_1(x, y) = 2bq \sin \alpha \, \mathrm{Re}\, \Phi_2(z) \,, \quad y > 0 \,, \tag{47}$$

$$T_2(x, y) = \frac{2\kappa_1}{\kappa_2} bq \sin \alpha \, \mathrm{Re}\, \Phi_2(z) \,, \quad y < 0 \,, \tag{48}$$

where $z = x + \mathrm{i}y$ and

$$\Phi_2(z) = \frac{1}{2\pi \mathrm{i}} \left(\int_{-b}^{-a} + \int_a^b \right) \frac{Z(A(t), k)}{t - z} \, \mathrm{d}t \,. \tag{49}$$

It is shown in Appendix 2 that

$$\Phi_2(z) = \frac{1}{2\mathrm{i}} \left[G(z) - \frac{z}{b} \right], \tag{50}$$

where

$$G(z) = -\frac{1}{K} z(z^2 - b^2)^{1/2}(z^2 - a^2)^{1/2} \int_a^b \frac{\mathrm{d}t}{(b^2 - t^2)^{1/2}(t^2 - a^2)^{1/2}(t^2 - z^2)} \,. \tag{51}$$

The function $G(z)$ as given above can be readily evaluated by numerical integration. Equations (47), (48) and (50) then yield the temperature distribution in the body as follows:

$$T_1'(x, y) = q[x \cos \alpha + b \sin \alpha \, \mathrm{Im}\, G(z)] \,, \quad y > 0 \,, \tag{52}$$

$$T_2'(x, y) = q\left[x \cos \alpha + \frac{\kappa_1}{\kappa_2} b \sin \alpha \, \mathrm{Im}\, G(z)\right], \quad y < 0 \,. \tag{53}$$

It should be mentioned that $G(z)$ is expressible in terms of the complex Jacobian zeta function, namely

$$G(z) = \mathrm{i}\, \mathrm{sgn}(y) Z(A(z), k)$$

and hence

$$T_1'(x, y) = q[x \cos \alpha + b \sin \alpha \, \mathrm{Re}\, Z(A(z), k)] \,, \quad y > 0 \,, \tag{54}$$

$$T_2'(x, y) = q\left[x \cos \alpha - \frac{\kappa_1}{\kappa_2} b \sin \alpha \, \mathrm{Re}\, Z(A(z), k)\right], \quad y < 0 \,, \tag{55}$$

Using the results given in [4, form. 115.01], it is easily shown that

$$\operatorname{Re} Z(A(z), k) = E(\beta, k) + \frac{k^2 \sin\beta \cos\beta \sin^2\gamma\,(1 - k^2 \sin^2\beta)^{1/2}}{\cos^2\gamma + k^2 \sin^2\beta \sin^2\gamma} - \frac{E}{K} F(\beta, k) \tag{56}$$

with

$$\sin A(z) = \left(\frac{b^2 - z^2}{b^2 - a^2}\right)^{1/2}, \qquad A(z) = \theta + \mathrm{i}\phi ,$$

where the square root $(b^2 - z^2)^{1/2}$ stands for the principal value i.e. $\operatorname{Re}(b^2 - z^2)^{1/2} > 0$; the parameters γ and β are related to θ and ϕ by the relations

$$\cosh\phi \sin\theta = \frac{\sin\beta\,(1 - k'^2 \sin^2\gamma)^{1/2}}{\cos^2\gamma + k^2 \sin^2\beta \sin^2\gamma},$$

$$\sinh\phi \cos\theta = \frac{\cos\beta \cos\gamma \sin\gamma\,(1 - k^2 \sin^2\beta)^{1/2}}{\cos^2\gamma + k^2 \sin^2\beta \sin^2\gamma},$$

$$k'^2 + k^2 = 1 .$$

The temperature distribution as given by (54) and (55) with (56) is less suitable for numerical computation. By letting $\kappa_1 = \kappa_2$, the results for the case of identical materials is recovered. It appears that there is an error in the sign of the result given by Sih [1, form. (29)].

7. Concluding remarks

The solutions $T_1(x, y)$ and $T_2(x, y)$ given by (25) and (26), respectively, for a bi-material body with line of cracks under uniform heat flow obtained here are basically identical with those of Sih's [1] solutions for a homogeneous infinite medium. The difference lies in the multiplicative constants $2\kappa_2/(\kappa_1 + \kappa_2)$ and $2\kappa_1/(\kappa_1 + \kappa_2)$. The temperature distribution in the body $T_1'(x, y)$, $y > 0$ is identical to that for a homogeneous body given by Sih [1], while the temperature distribution $T_2'(x, y)$, $y < 0$ differs from that of Sih [1] by a factor κ_1/κ_2.

Appendix 1

An alternate solution of the boundary value problems for T_1 and T_2, with boundary conditions (12)–(16), leading to (25) and (26) without the use of Fourier transform is given here.

The boundary value problem can be reduced to a Dirichlet problem and a Neumann problem for the Laplace equation in the half-plane $y > 0$. Introduce the linear combinations

$$U(x, y) = T_1(x, y) - T_2(x, -y), \quad y > 0, \tag{57}$$

$$V(x, y) = \kappa_1 T_1(x, y) + \kappa_2 T_2(x, -y), \quad y > 0, \tag{58}$$

then U and V must satisfy the Laplace equation in the half-plane $y > 0$, and the boundary conditions

$$U(x,0)=\begin{cases}0\,, & x\not\in L\\ \Delta T(x)\,, & x\in L\end{cases} \tag{59}$$

obtained from (12) and

$$\frac{\partial V}{\partial y}(x,0)=0 \quad \text{for all } x \tag{60}$$

from (17).

The solution of the Dirichlet problem for U is well known:

$$U(x,y)=\frac{y}{\pi}\int_L \frac{\Delta T(t)}{(x-t)^2+y^2}\,\mathrm{d}t\,, \quad y>0\,. \tag{61}$$

The solution of the Neumann problem for V is trivial:

$$V(x,y)=\kappa_1 T_1(x,y)+\kappa_2 T(x,-y)\equiv 0\,, \quad y\geqslant 0\,. \tag{62}$$

Equations (62) and (57) with (61) yield the solutions for T_1 and T_2 as follows:

$$T_1(x,y)=\frac{\kappa_2}{\kappa_1+\kappa_2}\,\frac{y}{\pi}\int_L \frac{\Delta T(t)}{(x-t)^2+y^2}\,\mathrm{d}t\,, \quad y>0\,, \tag{63}$$

$$T_2(x,y)=\frac{\kappa_1}{\kappa_1+\kappa_2}\,\frac{y}{\pi}\int_L \frac{\Delta T(t)}{(x-t)^2+y^2}\,\mathrm{d}t\,, \quad y<0\,. \tag{64}$$

Appendix 2

The Cauchy integrals $\Phi_1(z)$ and $\Phi_2(z)$, defined in (39) and (49) respectively, are determined here.

To evaluate $\Phi_1(z)$, consider the integral

$$I_1=\oint_C \frac{(w^2-a^2)^{1/2}}{w-z}\,\mathrm{d}w\,, \tag{65}$$

where C is the path shown in Fig. 3. The branch of the function $(w^2-a^2)^{1/2}$ is chosen in such a way that $(w^2-a^2)^{1/2}$ coincides with the positive square root if w is real and $>a$, and that the function $(w^2-a^2)^{1/2}$ is analytic in the complex plane with a branch cut along the segment $[-a,a]$. The integrals along the circle C_ε vanish as $\varepsilon\to 0$, while

$$\oint_{C_R} \frac{(w^2-a^2)^{1/2}}{w-z}\,\mathrm{d}w\to 2\pi \mathrm{i}z \quad \text{as } R\to\infty\,.$$

Application of the residue theorem then yields

$$\Phi_1(z)=\frac{1}{2\mathrm{i}}\,[(z^2-a^2)^{1/2}-z]\,. \tag{66}$$

To evaluate $\Phi_2(z)$, consider the general integral

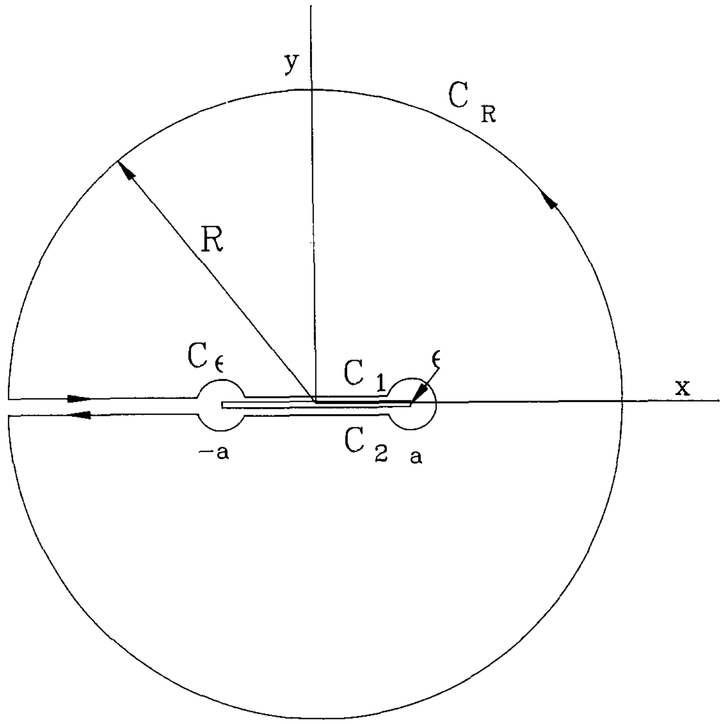

Fig. 3. Contour for the integral I_1.

$$I_2 = \oint_C \frac{G(w)}{w - z}\, \mathrm{d}w \,, \tag{67}$$

where C is the path shown in Fig. 4, and where the function $G(w)$ is required to be analytic in the complex plane with branch cuts $[-b, -a]$ and $[a, b]$. The integral can be evaluated either directly or using the residue theorem, with the result

$$I_2 = \left(\int_{-b}^{-a} + \int_a^b \right) \frac{G^+(t) - G^-(t)}{t - z}\, \mathrm{d}t + \lim_{R\to\infty} \oint_{C_R} \frac{G(z)}{w - z}\, \mathrm{d}w = 2\pi \mathrm{i} G(z) \,. \tag{68}$$

Here, $G^{\pm}(t) = G(t \pm \mathrm{i}0)$, $a < |t| < b$, denote the limit values of $G(w)$ on the upper and lower sides of the branch cuts.

The function $G(z)$ is chosen as follows:

$$G(z) = \frac{-1}{2K} (z^2 - a^2)^{1/2} (z^2 - b^2)^{1/2} \left(\int_{-a}^{-b} + \int_a^b \right) \frac{\mathrm{d}t}{(b^2 - t^2)^{1/2} (t^2 - a^2)^{1/2} (t - z)} \,, \tag{69}$$

where the square roots $(z^2 - b^2)^{1/2}$ and $(z^2 - a^2)^{1/2}$ are defined in the usual manner. In (69), the product $(z^2 - b^2)^{1/2}(z^2 - a^2)^{1/2}$ and the integral that follows are analytic in the complex plane with branch cuts $[-b, -a]$ and $[a, b]$. Thus, the function $G(z)$ chosen is analytic except for branch cuts $[-b, -a]$ and $[a, b]$. The behavior of $G(z)$ as $z \to \infty$ is found to be

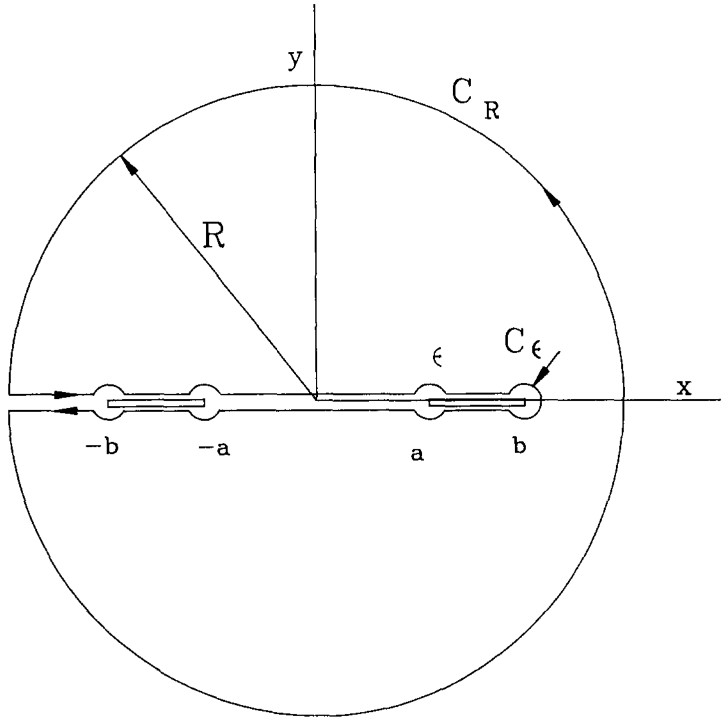

Fig. 4. Contour for the integral I_2.

$$G(z) = -\frac{1}{2K} z^2[1 + O(z^{-2})]\left[-\left(\int_{-b}^{-a} + \int_a^b\right) \frac{\mathrm{d}t}{(b^2 - t^2)^{1/2}(t^2 - a^2)^{1/2}} z^{-1} + O(z^{-3})\right]$$
$$= \frac{z}{b} + O(z^{-1}), \quad (z \to \infty).$$

From this result it immediately follows that

$$\lim_{R\to\infty} \oint_{C_R} \frac{G(w)}{w - z} \,\mathrm{d}w = 2\pi\mathrm{i} \frac{z}{b}. \tag{70}$$

Next, the limit values $G^{\pm}(x) = G(x \pm \mathrm{i}0)$, $a < |x| < b$, are determined. By means of Plemelj's formulas it is found that

$$G^+(x) - G^-(x)$$
$$= -\frac{\mathrm{i}}{K} \operatorname{sgn}(x)(b^2 - x^2)^{1/2}(x^2 - a^2)^{1/2}\left(\int_{-b}^{-a} + \int_a^b\right) \frac{\mathrm{d}t}{(b^2 - t^2)^{1/2}(t^2 - a^2)^{1/2}(t - x)}$$
$$= -\frac{2\mathrm{i}}{K} |x|(b^2 - x^2)^{1/2}(x^2 - a^2)^{1/2} \int_a^b \frac{\mathrm{d}t}{(b^2 - t^2)^{1/2}(t^2 - a^2)^{1/2}(t^2 - x^2)}$$
$$= 2\mathrm{i}Z(A(x), k), \quad a < |x| < b, \tag{71}$$

where the integral after (45) was used.

Finally, the substitution of (70) and (71) into (68) yields the following result:

$$\Phi_2(z) = \frac{1}{2\pi \mathrm{i}} \left(\int_{-b}^{-a} + \int_{a}^{b} \right) \frac{Z(A(t), k)}{t - z} \, \mathrm{d}t = \frac{1}{2\mathrm{i}} \left[G(z) - \frac{z}{b} \right]. \tag{72}$$

Acknowledgement

The author would like to express his sincere appreciation to Professor J. Boersma of the Department of Mathematics, Eindhoven University of Technology, The Netherlands, for his valuable comments and suggestions which have made the mathematical analysis of the paper sound. In particular, the analysis of Section 4 of the paper has been improved considerably and the evaluation of the contour integrals, as shown in the Appendix 2, has become more rigorous. Professor Boersma's contribution to the paper is considerable and his name could have been listed as one of the authors.

References

1. G. Sih, Heat Conduction in Infinite Medium With Lines of Discontinuities. *Journal of Heat Transfer*, Transactions of ASME 87 (1965) 293–298.
2. N.I. Muskhelishvili, *Singular Integral Equations*. Groningen: P. Noordhoff N.V. (1958) 435 pp.
3. I.N. Sneddon, *Fourier Transforms*. New York: McGraw-Hill Co. (1951) 542 pp.
4. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Tables of Integral Transforms*, Vol. II. New York: McGraw-Hill (1954) 451 pp.
5. P.F. Byrd and M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*. New York: Springer-Verlag (1971) 358 pp.